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Sampling properties of estimators of the log-logistic distribution with application to Canadian precipitation data

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ABSTRACT

We consider the probability-weighted moment and the maximum-likelihood estimators of two parameters in the log-logistic distribution. Quantile estimators are obtained using both methods. The distributional properties of these estimators are studied in large samples, via asymptotic theory, and in small and moderate samples, via Monte Carlo simulation. The distribution is shown to be appropriate for a wide variety of meteorological data.

RÉSUMÉ

On considère l'estimateur des moments pondérés par les probabilités et l'estimateur du maximum de vraisemblance des deux paramètres d'une loi log-logistique. On obtient des estimateurs des quantiles par les deux méthodes. Les propriétés asymptotiques des distributions de ces estimateurs sont étudiées pour les échantillons de grande taille. Pour les échantillons de petite taille, on emploie une simulation Monte Carlo. On montre que la distribution s'applique à une grande variété de données météorologiques.

1. INTRODUCTION

Systems of distributions are generally generated to produce forms of parametric families useful for data analysis. The principal aim in choosing one of these forms of distributions is to facilitate the mathematical analysis to which it is subjected, while attaining a reasonable approximation. One way of generating such systems is achieved by employing simple transformations on some of the well-known distributions. The lognormal, logit normal, and the $\sinh^{-1}$ normal systems of frequency curves are obtained by three simple transformations of the normal curve (Tadikamalla and Johnson 1982). The logarithmic transformation can be applied to the logistic distribution to obtain the log-logistic distribution (LLD). Thus, if z is a random variable (r.v.) which has a standard logistic distribution with probability density function $g(z) = e^z(1 + e^z)^{-2}$, then using the transformation $Z = \beta \ln (T/\alpha)$, the probability density function of T will be expressed as

$$f(t) = \frac{(\beta/\alpha)(t/\alpha)^{\beta-1}}{\{1 + (t/\alpha)^\beta\}^2}, \qquad \alpha, t > 0, \quad \beta \geq 1. \tag{1.1}$$

A random variable T whose probability density function is given by (1.1) is said to have an LLD. This distribution is a special case of Burr's type-XII distribution (Burr 1942).

The LLD is also a special case of the "kappa distributions" introduced by Mielke and Johnson (1973), which were shown to give a reasonably good fit to realized sets of precipitation and stream-flow data. The distribution is also obtained by compounding the three-parameter Weibull distribution

$$f(t|P) = P(\beta/\alpha)(t/\alpha)^{\beta-1}\, e^{-P(x/\alpha)^{\beta}} \tag{1.2}$$

over the probability distribution of P, which is taken as standard exponential. That is,

$$f(t) = \int_0^{\infty} f(t|P)\, e^{-P}\, dP.$$

A characterization of the LLD will be given in the Appendix. The LLD is very similar in shape to the lognormal distribution (Figures 1 and 2) but is mathematically more tractable and hence very useful in a wide variety of applications, especially analysis of survival data (Bennett 1983). It is also noted that the LLD is a good alternative to the Weibull, whose hazard function, while it may be increasing or decreasing, must be monotonic, whatever the values of its parameters. This may be inappropriate where the course of the disease is such that mortality reaches a peak after some finite period, and then slowly declines. The main objective of this work is to estimate the parameters of the LLD and to fit the distribution to precipitation data collected over a number of years for some Canadian cities. In this article we introduce a new class of estimators, known as *probability-weighted moment* (PWM) estimators, discuss some of their asymptotic properties, and compare their performance with maximum-likelihood (ML) estimators. Small-sample assessments and data analysis are presented subsequently.

2. PROBABILITY WEIGHTED MOMENTS

A new class of moments, called probability-weighted moments (PWM), was introduced by Greenwood *et al.* (1979) and later applied by Hosking, Wallis, and Wood (1985) to the extreme-value distribution. It is realized that the method of PWM is of potential interest for populations whose cumulative distribution function can be written in an inverse form; that is, if T is a r.v. and $F(t) = P(T \le t)$, then $t = t(F)$. These distributions include, among others, the extreme-value, the Weibull, the Burr, Tukey's lambda, Mielke and Johnson's kappa, and the LLD. Greenwood *et al.* (1979) defined the PWM as

$$W_{l,j,k} = \mathscr{E}[T^l F^j (1-F)^k] \tag{2.1}$$

where l, j, k are real numbers. For $l = 1$ and $k = 0$, $W_r = W_{1,r,0} = \mathscr{E}[T\{F(T)\}^r]$ will denote the PWM of order r. Since for the LLD

$$F(t) = \int_0^t f(x)\, dx = \frac{(t/\alpha)^{\beta}}{1 + (t/\alpha)^{\beta}},$$

then

$$\begin{aligned} W_r &= \beta \int_0^{\infty} \frac{[(t/\alpha)^{\beta}]^{r+1}}{[1 + (t/\alpha)^{\beta}]^{r+2}}\, dt \\ &= \frac{\alpha\Gamma(r + 1 + 1/\beta)\Gamma(1 - 1/\beta)}{\Gamma(r + 2)}, \qquad r = 0, 1, 2, \ldots, \quad \beta > 1. \end{aligned} \tag{2.2}$$

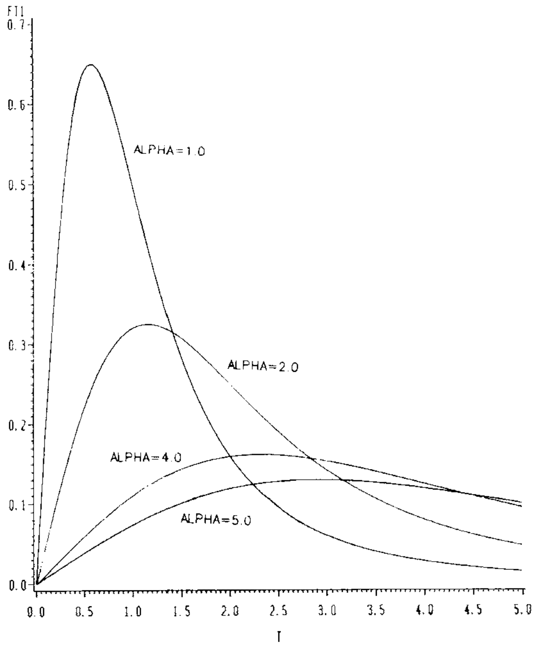

FIGURE 1: Graph of the probability density function of the LLD for $\beta = 2.0$.

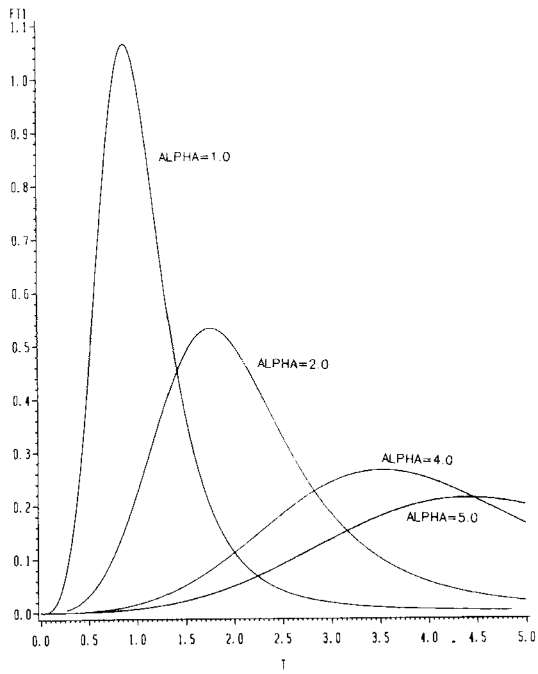

FIGURE 2: Graph of the probability density function of the LLD for $\beta = 4.0$.

From (2.2) we have

$$W_0 = \alpha\Gamma\left(1 + \frac{1}{\beta}\right)\Gamma\left(1 - \frac{1}{\beta}\right) = \frac{\alpha\pi}{\beta}\left(\sin\frac{\pi}{\beta}\right)^{-1}, \tag{2.3}$$

$$W_1 = \frac{1 + \beta}{2\beta} W_0. \tag{2.4}$$

Given a random sample of size n from the LLD, estimation of W_r is most conveniently based on the order sample $t_{(1)} < t_{(2)} < \cdots < t_{(n)}$. The statistic

$$\hat{W}_r = n^{-1} \sum_{j=1}^{n} t_{(j)} \prod_{i=1}^{r} \frac{j - i}{n - i} \tag{2.5}$$

in an unbiased estimator of W_r (Landwehr, Matalas, and Wallis 1979). The PWM estimators α^*, β^* of the parameters are the solutions of (2.3), (2.4) for α and β when W_r are replaced by their estimators $\hat{W}_r$ (2.5). Thus,

$$\beta^* = \frac{\hat{W}_0}{2\hat{W}_1 - \hat{W}_0} \tag{2.6}$$

and

$$\alpha^* = \frac{\hat{W}_0^2 \sin(\pi/\beta^*)}{\pi(2\hat{W}_1 - \hat{W}_0)} \tag{2.7}$$

are the PWM estimators of the parameters β and α respectively.

3. ASYMPTOTIC RESULTS

It is impossible, at least analytically, to estimate the required sample size that ensures the applicability of the large-sample theory in inference. However, it is quite important to investigate the first-order asymptotes of the PWM estimators. In particular, comparisons will be made with respect to asymptotic biases and variances of the ML estimators. We consider first the asymptotic distribution of $\hat{W}_r$. As can be seen from (2.5), $\hat{W}_r$ is a linear combination of the order statistics $t_{(1)} < t_{(2)} < \cdots < t_{(n)}$, and the results of Chernoff, Gastwirth, and Johns (1967) will be used to establish that $(\hat{W}_0, \hat{W}_1)^T$ has, asymptotically, bivariate normal distribution with mean $(W_0, W_1)^T$ and variance-covariance matrix $(1/n)$ $\{\sigma_{ij}\}$. Following the method of Hosking, Wallis, and Wood (1985), it can be shown that

$$\sigma_{rs} = \tfrac{1}{2}(\phi_{rs} + \phi_{sr}), \qquad r, s = 0, 1, \ldots, \tag{3.1}$$

where

$$\phi_{rs} = 2\int_0^\infty \int_0^y \{F(x)\}^{r+1}\{F(y)\}^s\{1 - F(y)\}\, dx\, dy. \tag{3.2}$$

For the LLD, one can show that

$$\phi_{rs} = \frac{2\alpha^2}{\beta\{1 + (r + 1)\beta\}}\left\{B\left(r + s + \frac{2}{\beta} + 1, 1 - \frac{2}{\beta}\right) + \sum_{n=1}^{\infty} \frac{B(n, r + 1/\beta + 2)}{B(n, r + 1)} B\left(n + r + s + \frac{2}{\beta} + 1, 1 - \frac{2}{\beta}\right)\right\}, \qquad \beta > 2. \tag{3.3}$$

Hence

$$\sigma_{00} = \frac{2\alpha^2\Gamma(1 - 2/\beta)}{\beta(1 + \beta)} \left\{\Gamma\left(1 + \frac{2}{\beta}\right) + \Gamma\left(2 + \frac{1}{\beta}\right) \sum_{n=1}^{\infty} \frac{\Gamma(n + 1 + 2/\beta)}{(n + 1)\Gamma(n + 2 + 1/\beta)}\right\}, \quad (3.4)$$

$$\sigma_{10} = \sigma_{01} = \frac{\alpha^2\Gamma(1 - 2/\beta)}{\beta(1 + \beta)(1 + 2\beta)} \left\{\frac{(2 + 3\beta)\Gamma(2 + 2/\beta)}{2} + (1 + 2\beta)\Gamma\left(2 + \frac{1}{\beta}\right) \sum_{n=1}^{\infty} \frac{\Gamma(n + 2 + 2/\beta)}{(n + 1)(n + 2)\Gamma(n + 2 + 1/\beta)} + (1 + \beta)\Gamma\left(3 + \frac{1}{\beta}\right) \sum_{n=1}^{\infty} \frac{\Gamma(n + 2 + 2/\beta)}{(n + 2)\Gamma(n + 3 + 1/\beta)}\right\}, \quad (3.5)$$

$$\sigma_{11} = \frac{2\alpha^2\Gamma(1 - 2/\beta)}{\beta(1 + 2\beta)} \times \left\{\frac{\Gamma(3 + 2/\beta)}{6} + \Gamma\left(3 + \frac{1}{\beta}\right) \sum_{n=1}^{\infty} \frac{\Gamma(n + 3 + 2/\beta)}{(n + 2)(n + 3)\Gamma(n + 3 + 1/\beta)}\right\}. \quad (3.6)$$

The biases, variances, and covariances of the PWM can then be obtained using the delta method given by Kendall and Stuart (1977). The expressions are quite complicated and long and will not be included.

For the purpose of comparison, asymptotic biases of the ML estimators of the parameters are calculated using the technique given by Shenton and Bowman (1977), which, on omitting the heavy algebra involved, gives

$$\text{bias}(\hat{\alpha}) \simeq n^{-1} \frac{1.5\alpha}{\beta^2}, \qquad \text{bias}(\hat{\beta}) \simeq n^{-1} \left(\frac{6.75\beta}{\pi^2 + 3} + \frac{2.25\beta(6 + 5\pi^2)}{(\pi^2 + 3)^2}\right). \quad (3.7)$$

The asymptotic variances of the ML estimators can be obtained from the inverse of Fisher's information matrix and are given by

$$\mathcal{V}ar(\hat{\alpha}) \simeq n^{-1} \frac{3\alpha^2}{\beta^2}, \qquad \mathcal{V}ar(\hat{\beta}) \simeq n^{-1} \frac{9\beta^2}{(\pi^2 + 3)}, \qquad \mathcal{C}ov(\hat{\alpha}, \hat{\beta}) \simeq 0. \quad (3.8)$$

Graphs of the asymptotic biases of the PWM and the ML estimators of α and β are in Figures 3 and 4, respectively. For $\beta \leq 2$ these expectations do not exist, as demonstrated by expressions (3.4), (3.5), (3.6) and the graphs. Biases of both estimators of the parameter α are scale-invariant, and hence can be obtained for any value of α by direct multiplication. For large values of β, the estimators of the parameter α are asymptotically unbiased. For the parameter β, the PWM estimator is asymptotically less biased than the ML estimator for $\beta \geq 4$. The asymptotic relative efficiencies are shown in Figure 5. Asymptotic efficiency is defined as the ratio

$$\text{EFF}(\overset{*}{\theta}) = \lim_{n \to \infty} \frac{\mathcal{V}ar(\hat{\theta})}{\mathcal{V}ar(\overset{*}{\theta})}$$

for each element of the parameter vector, where $\hat{\theta}$ is the ML estimator of θ. For values of $\beta > 7$ the PWM estimators seem to be reasonably efficient; each parameter estimator has efficiency of more than 90%.

Similar results may be obtained for ML and PWM estimators of the quantiles of the

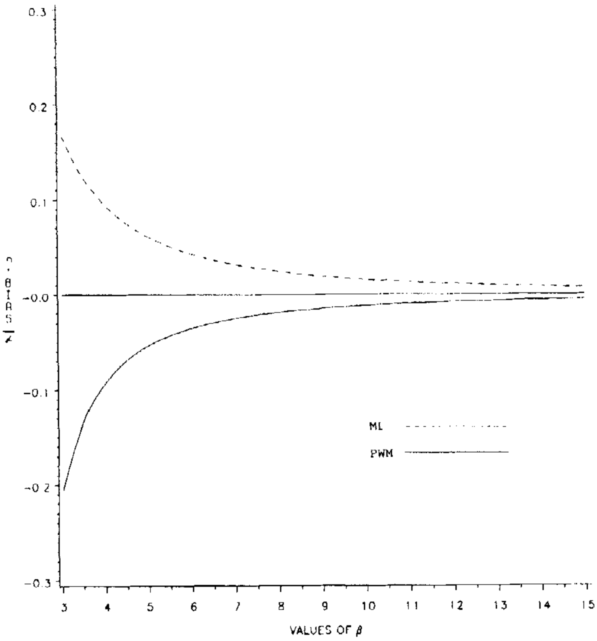

FIGURE 3: Asymptotic biases of PWM and ML estimators for the parameter α.

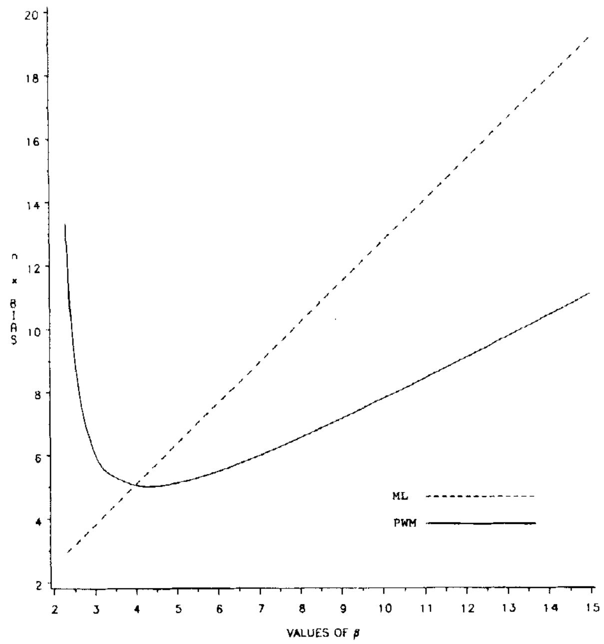

FIGURE 4: Asymptotic biases of PWM and ML estimators for the parameter β.

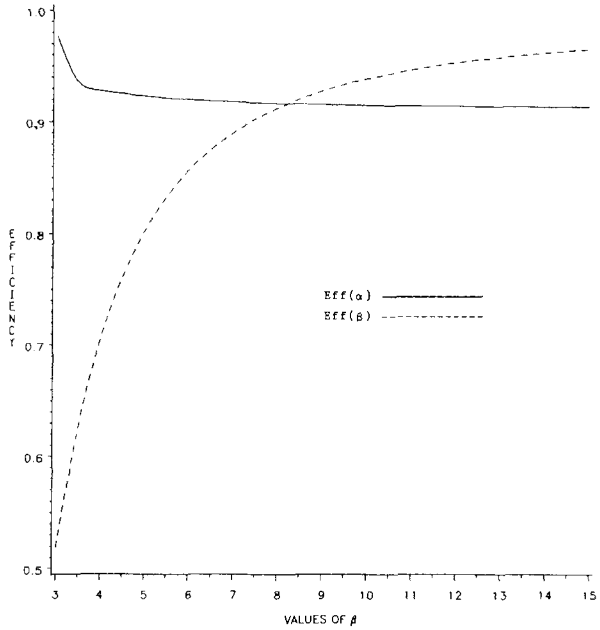

FIGURE 5: Asymptotic relative efficiency of the PWM estimators for the parameters α and β.

LLD. The quantiles are given in terms of the parameters by

$$t(F) = \alpha \left(\frac{F}{1-F}\right)^{1/\beta}. \tag{3.9}$$

A quantile estimator $\tilde{t}(F)$ is defined by substituting estimators $\tilde{\alpha}$ and $\tilde{\beta}$ of the parameters α and β respectively in (3.9). The variance of $\tilde{t}(F)$ is given asymptotically by

$$\mathcal{V}ar(\tilde{t}(F)) \simeq \left(\frac{F}{1-F}\right)^{2/\beta} \mathcal{V}ar(\tilde{\alpha}) - 2\,\frac{\alpha}{\beta^2}\left(\frac{F}{1-F}\right)^{2/\beta} \ln\left(\frac{F}{1-F}\right) \mathcal{C}ov(\tilde{\alpha}, \tilde{\beta})$$
$$+ \left\{\frac{\alpha}{\beta^2}\left(\frac{F}{1-F}\right)^{1/\beta} \ln\left(\frac{F}{1-F}\right)\right\}^2 \mathcal{V}ar(\tilde{\beta}). \tag{3.10}$$

In the case of the PWM estimator of the quantile, the variance may be obtained by reading α^*, β^* for $\tilde{\alpha}$, $\tilde{\beta}$ in (3.10). Similarly, for the ML estimator, we read $\hat{\alpha}$, $\hat{\beta}$ for $\tilde{\alpha}$, $\tilde{\beta}$, noting that $\mathcal{C}ov(\hat{\alpha}, \hat{\beta}) = 0$. The expressions for the variances are not presented here, but the results are summarized in Figures 6 and 7 for the asymptotic relative efficiency of the quantile estimators. The PWM quantile estimator has poor efficiency in the extreme tails of the distribution, especially for $\beta < 4$. However, the efficiency in the extreme lower tail seems to be higher than in the extreme upper tail.

4. SMALL-SAMPLE ASSESSMENT

The assessment of the small-sample properties of the PWM and ML estimators was achieved by computer simulation of data for combinations of values of (α, β) and sample sizes $n = 15, 25$. For each combination of values, 1000 samples were generated from the LLD, using IMSL (1987) subroutine RNGCT (Stat/Library), and for each sample the

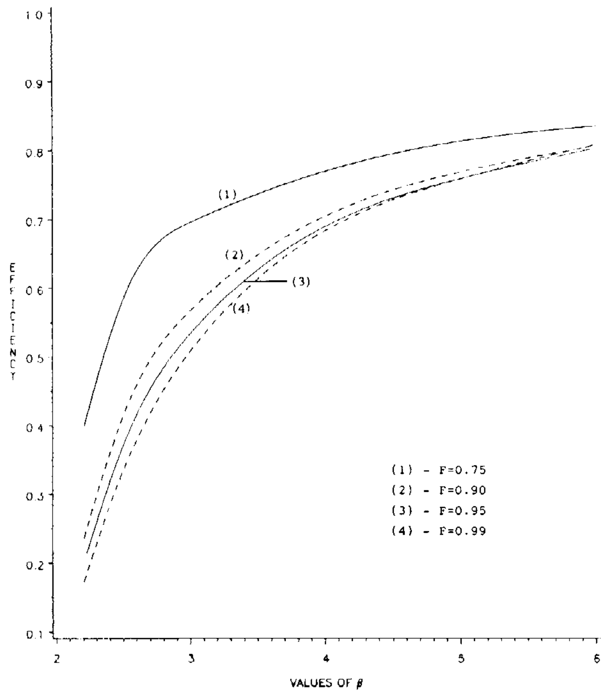

FIGURE 6: Asymptotic relative efficiency of the PWM estimators for quantiles.

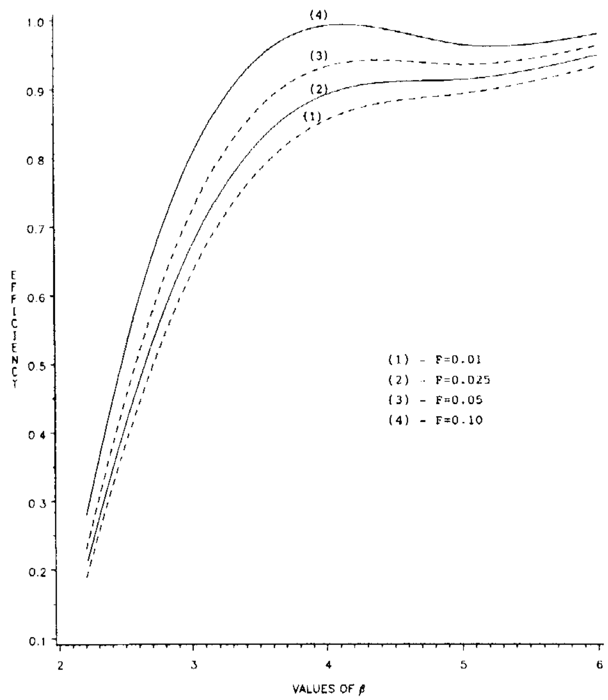

FIGURE 7: Asymptotic relative efficiency of the PWM estimators for quantiles.

TABLE 1: Failure rate of ML estimation for the log-logistic distribution[a]

n	α	$\beta = 2.0$	4.0	6.0
15	1.0	0.0282	0.1597	0.2877
	5.0	0.0010	0.0375	0.1304
	10.0	0.0010	0.0338	0.0926
25	1.0	0.0357	0.2156	0.3026
	5.0	0.0000	0.0329	0.0842
	10.0	0.0010	0.0385	0.0440

[a]Tabulated values are based on the number of failures to obtain 1000 satisfactory runs.

parameters α and β were estimated by the method of PWM and the ML method using the IMSL (1987) subroutine BCONG (Math/Library) to maximize the likelihood function. We used α^* and β^* as starting points for the maximization of the likelihood function. The log-likelihood function for a simple random sample $(t_1, \ldots, t_n)$ is

$$l = n \ln \frac{\beta}{\alpha} + (\beta - 1) \sum_{i=1}^{n} \ln u_i - 2 \sum_{i=1}^{n} \ln (1 + u_i^{\beta}),$$

where $u_i = t_i/\alpha$.

The BCONG subroutine uses a quasi-Newton method with bounds on the variables ($\alpha > 0$, $\beta > 1$). The gradient and tolerance limits 10^{-6} were supplied to the routine. For some samples, however, it appears that l does not have a local maximum. Whenever a sample is simulated from the LLD, BCONG is called to locate the ML estimates, and if the optimization criteria are not met, this sample is discarded and is replaced by a new generated sample. The process continues till 1000 samples are available, each of which gives ML estimates. Table 1 gives the fraction of samples, $n'/(1000 + n')$, for which BCONG failed to find its local maximum. The occurrence of such samples is found to be rare when α and n are increased.

One of the characteristics of the precipitation data that have been analyzed in Section 5 is that α and n are large. Therefore, the location of ML estimates was successfully completed in a small number of iterations.

The simulation results are summarized in Tables 2, 3, 4, and 5. Results for the shape

TABLE 2: Small-sample biases of probability-weighted moment and maximum-likelihood estimators for α

		$\alpha = 1.0$		5.0		10.0	
n	β	PWM	ML	PWM	ML	PWM	ML
15	2.0	−0.0187	0.2311	−0.1013	0.2658	−0.1771	0.2569
	4.0	−0.0121	0.1893	−0.0236	0.2057	−0.1302	0.2907
	6.0	−0.0049	0.2233	0.0047	0.2059	−0.0044	0.2880
	8.0	−0.0085	0.1671	−0.0302	−0.1152	−0.0863	0.0143
	10.0	−0.0112	−0.6842	−0.0333	−0.9092	−0.0194	−0.8717
25	2.0	−0.0013	0.1860	−0.0473	0.1957	−0.0904	0.2169
	4.0	0.0013	0.0775	−0.0227	0.1534	−0.1249	0.1687
	6.0	−0.0043	−0.0265	−0.0275	0.1657	−0.0329	0.1921
	8.0	−0.0109	−0.1570	−0.0345	−0.0310	−0.0467	−0.0387
	10.0	−0.0113	−0.7659	−0.0266	−1.0951	−0.0663	−1.1823

TABLE 3: Small-sample biases of probability-weighted moment and maximum-likelihood estimators for β

		$\alpha = 1.0$		5.0		10.0	
n	β	PWM	ML	PWM	ML	PWM	ML
15	2.0	0.0208	0.1119	0.0756	0.1888	0.2148	0.1667
	4.0	0.0022	0.2396	0.0404	0.2883	0.0001	0.3592
	6.0	0.0034	0.3466	0.0282	0.4181	0.0479	0.4826
	8.0	−0.0024	0.3191	−0.0052	0.1136	−0.0316	0.2265
	10.0	−0.0054	−0.5782	0.0179	−0.7337	0.0857	−0.7162
25	2.0	0.0236	0.0568	0.0672	0.1122	0.1393	0.1204
	4.0	0.0096	0.0870	0.0124	0.1966	−0.0373	0.1856
	6.0	0.0025	0.0082	−0.0073	0.2583	−0.0009	0.3028
	8.0	−0.0052	−0.1383	−0.0136	0.0512	−0.0075	0.0510
	10.0	−0.0059	−0.8081	0.0264	−1.1258	0.0327	−1.1860

TABLE 4: Small-sample biases of probability-weighted moment and maximum-likelihood estimators for α

		$\alpha = 1.0$		5.0		10.0	
n	β	PWM	ML	PWM	ML	PWM	ML
15	2.0	0.0478	0.3109	1.2122	0.2975	5.3655	0.3106
	4.0	0.0143	1.0363	0.3142	0.9235	1.3142	1.1076
	6.0	0.0067	2.3866	0.1554	2.0708	0.5611	2.1230
	8.0	0.0042	3.6198	0.0850	3.3091	0.3594	3.6673
	10.0	0.0028	3.1151	0.0756	5.7859	0.2983	4.6686
25	2.0	0.0318	0.1806	0.7656	0.1855	2.9386	0.1831
	4.0	0.0090	0.5736	0.1908	0.5619	0.7764	0.6101
	6.0	0.0039	1.1901	0.0933	1.3131	0.3702	1.1768
	8.0	0.0022	1.8280	0.0532	1.8435	0.2081	1.8269
	10.0	0.0017	2.0124	0.0450	2.4072	0.1953	2.4699

parameter β are of greater importance, since this parameter determines the overall shape of the LLD and the rate of increase of the upper quantiles $t(F)$ as F approaches 1. In general the PWM estimators of both parameters seem to be less biased and almost consistently have smaller variance. The variances of the PWM estimators decrease when β increases, while the ML estimators increase with β.

5. DATA ANALYSIS

Mielke and Johnson (1973) found that the kappa 3 distribution, which is a member of Burr's XII family, compares very favourably with the gamme and lognormal distributions in being able to describe natural precipitation data. Because the LLD is a special case of Burr's family, we evaluate its goodness of fit as a chance mechanism generating precipitation data obtained from the Canadian Climate Centre collected from five Canadian cities over a period of several years.

For each year, the Canadian data profiles have two types of precipitation measurements (centimetres of water) for each region: total precipitation (TP) for each year, and maximum precipitation (MP) in 24 hours. Table 6 summarizes the data by region, type and period.

TABLE 5: Small-sample biases of probability-weighted moment and maximum-likelihood estimators for β

		α = 1.0		5.0		10.0	
n	β	PWM	ML	PWM	ML	PWM	ML
15	2.0	0.0514	0.3103	1.2756	0.2883	5.8132	0.3009
	4.0	0.0140	1.0855	0.3097	0.9173	1.3111	1.1146
	6.0	0.0065	2.5205	0.1463	2.1530	0.5307	2.2543
	8.0	0.0042	3.7586	0.0811	3.6386	0.3384	3.9142
	10.0	0.0030	3.2488	0.0721	6.4104	0.2751	5.1467
25	2.0	0.0333	0.1512	0.7911	0.1474	3.0375	0.1537
	4.0	0.0086	0.5616	0.1816	0.5316	0.7573	0.5478
	6.0	0.0037	1.2253	0.0868	1.3037	0.3603	1.1724
	8.0	0.0023	1.8465	0.0525	1.8639	0.2012	1.8746
	10.0	0.0018	2.0202	0.0412	2.5682	0.1787	2.7240

TABLE 6: Region by period for the TP and MP data

Region	Period	Sample size
Montreal	1942–1980	39
Regina	1905–1980	76
Shearwater	1945–1980	36
Toronto	1901–1980	80
Vancouver	1938–1980	43

Table 7 gives the summary of the data analysis based on the LLD. Except for the MP data collected from the Toronto area, the P-values based on the χ^2 statistic are higher than 10%.

6. CONCLUSIONS

Two methods of estimation for the parameters and the quantiles of LLD have been derived using the PWM and the ML methods. The PWM are fast and straightforward to compute and always yield feasible values for the estimated parameters. The biases and variances of the estimators seem to be small in comparison with those of the ML estimators, even when sample sizes as small as 15 or 25 observations are used. The efficiency of the PWM estimator of the parameter α is extremely high for all values of β. However, the efficiency of the PWM estimator of β is seriously low for values of β smaller than 6.

In analyzing the extensive data collected from the various Canadian regions, there was not sufficient evidence to suspect the suitability of the LLD as a model generating this type of data. Because its distribution function has a closed form, and its hazard function is quite flexible, the LLD is more appealing and may have wide variety of applications in many disciplines.

APPENDIX. CHARACTERIZATION OF THE LOG-LOGISTIC DISTRIBUTION

If a random variable T has an absolutely continuous (with respect to Lebesgue measure)

TABLE 7: Goodness-of-fit tests for LL Distribution on TP[a] and MP[b] data at different locations in Canada

		PWM method		ML method		
		Estimates (S.E.)	*P*-value	Estimates (S.E.)	*P*-value	Kernel of *l*
Montreal	TP	α^* = 943.1658(20.3022) β^* = 13.4807(1.8424)	>0.10	$\hat\alpha$ = 940.4094(19.5085) $\hat\beta$ = 13.3697(1.7903)	>0.10	−242.7385
	MP	α^* = 44.8938(1.9135) β^* = 6.7916(0.9667)	>0.10	$\hat\alpha$ = 45.0294(1.8009) $\hat\beta$ = 6.9349(0.9286)	>0.10	−151.0906
Regina	TP	α^* = 368.8494(10.8257) β^* = 7.0672(0.7176)	>0.10	$\hat\alpha$ = 371.4697(10.8958) $\hat\beta$ = 6.7736(0.6498)	>0.10	−453.9678
	MP	α^* = 36.2557(1.8658) β^* = 4.0050(0.4571)	>0.10	$\hat\alpha$ = 35.9319(1.7076) $\hat\beta$ = 4.1807(0.4010)	>0.10	−316.8103
Shearwater	TP	α^* = 1364.0918(39.2688) β^* = 10.4859(1.5049)	>0.10	$\hat\alpha$ = 1372.8477(38.7170) $\hat\beta$ = 10.2360(1.4267)	>0.10	−247.1965
	MP	α^* = 64.4835(3.7660) β^* = 5.1467(0.7957)	>0.10	$\hat\alpha$ = 63.2263(3.1883) $\hat\beta$ = 5.7246(0.7979)	>0.10	−159.6321
Toronto	TP	α^* = 789.6057(11.2565) β^* = 14.2129(1.3545)	>0.10	$\hat\alpha$ = 789.8479(10.6700) $\hat\beta$ = 14.4160(1.3478)	>0.10	−481.1758
	MP	α^* = 45.3150(1.3686) β^* = 6.6915(0.6662)	>0.05	$\hat\alpha$ = 44.6613(1.2479) $\hat\beta$ = 6.9303(0.6480)	>0.05	−310.1741
Vancouver	TP	α^* = 1072.1152(25.2318) β^* = 11.7393(1.5346)	>0.10	$\hat\alpha$ = 1084.0911(24.6059) $\hat\beta$ = 11.6373(1.4841)	>0.10	−280.6655
	MP	α^* = 43.9040(1.9299) β^* = 6.2681(0.8581)	>0.10	$\hat\alpha$ = 43.5020(1.7583) $\hat\beta$ = 6.5350(0.8334)	>0.10	−167.7991

[a]Total precipitation of the year.
[b]Maximum precipitation in 24 hours.

distribution function $F(t, \alpha, \beta)$, $t \geq 0$, $\beta > 0$, $\alpha > 0$, then T follows an LLD (1.1) if and only if for some real numbers $\xi > 1$, $\eta > 1$

$$P[1 + (T/\alpha)^{\beta} > \xi\eta] = P[1 + (T/\alpha)^{\beta} > \xi]P[1 + (T/\alpha)^{\beta} > \eta].$$

Proof. Condition is necessary: Let $T \sim$ LLD. Thus

$$P[1 + (T/\alpha)^{\beta} > \xi\eta] = 1 - F[\alpha(\xi\eta - 1)^{1/\beta}].$$

Since

$$F(t, \alpha, \beta) = \frac{(t/\alpha)^{\beta}}{1 + (t/\alpha)^{\beta}},$$

therefore

$$P[1 + (T/\alpha)^{\beta} > \xi\eta] = (\xi\eta)^{-1} = P[1 + (T/\alpha)^{\beta} > \xi]P[1 + (T/\alpha)^{\beta} > \eta].$$

Condition is sufficient: Suppose that T is a continuous random variable whose distribution function is $F(t, \alpha, \beta)$ and that

$$P[1 + (T/\alpha)^{\beta} > \xi\eta] = P[1 + (T/\alpha)^{\beta} > \xi]P[1 + (T/\alpha)^{\beta} > \eta].$$

Let

$$\psi(u) = P[1 + (T/\alpha)^{\beta} > u], \qquad t > 0.$$

Accordingly, $\psi(\xi\eta) = \psi(\xi)\psi(\eta)$.

A solution of the above functional equation was given by Aczel (1966) as

$$\psi(t) = at^{b}.$$

It can be shown that $\psi(t)$ is differentiable only if $a = 1$. Therefore

$$P[1 + (X/\alpha)^{\beta} > t] = t^{b},$$

or

$$1 - P[X < \alpha(t - 1)^{1/\beta}] = t^{b}.$$

Putting

$$\alpha(t - 1)^{1/\beta} = x,$$

then

$$1 - F(x, \alpha, \beta) = [1 + (x/\alpha)^{\beta}]^{b},$$

i.e.,

$$F(x, \alpha, \beta) = 1 - [1 + (x/\alpha)^{\beta}]^{b}.$$

Since $0 \leq x < \infty$, then $F(0, \alpha, \beta) = 0$, and for $F(\infty, \alpha, \beta) = 1$ we must have $b < 0$. Taking $b = -1$, we get

$$F(x, \alpha, \beta) = \frac{(x/\alpha)^{\beta}}{1 + (x/\alpha)^{\beta}}.$$

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